NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

TECHNICAL NOTE 2892

A RAPID METHOD FOR ESTIMATING THE SEPARATION POINT OF A

COMPRESSIBLE LAMINAR BOUNDARY LAYER

By Laurence K. Loftin, Jr., and Homer B. Wilson, Jr.

Langley Aeronautical Laboratory
Langley Field, Va.

NACA

Washington

February 1953

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2892

A RAPID METHOD FOR ESTIMATING THE SEPARATION POINT OF A

COMPRESSIBLE LAMINAR BOUNDARY LAYER

By Laurence K. Loftin, Jr., and Homer B. Wilson, Jr.

SUMMARY

A method has been developed for rapidly estimating the separation point of a laminar boundary layer in a compressible flow. The method consists of an extension of Von Doenhoff's simplified solution for the incompressible case (NACA TN 671) and makes use of a set of transforms derived by Stewartson (Proc. Roy. Soc., 1949) which permit, under certain assumptions, the expression of compressible laminar layers in terms of equivalent incompressible laminar layers. The method developed is generally applicable to any two-dimensional flow in which the classical boundary-layer assumptions are satisfied. The dependence of the method upon the boundary-layer assumptions, of course, means that it should not be applied to determine whether the pressure rise through a shock wave causes separation.

Calculations of the laminar separation point for a wide range of Mach number and velocity gradient indicate that, for all velocity gradients, the amount of velocity recovery possible before laminar separation occurs decreases as the Mach number increases.

INTRODUCTION

The determination of the position at which the laminar boundary layer separates has been the subject of much theoretical investigation. For the case of incompressible flow, such methods as those of Howarth (ref. 1), Von Kármán and Millikan (ref. 2), and Von Doenhoff's simplified adaptation of the Von Kármán-Millikan theory (ref. 3) may be used for determining the laminar separation point on bodies with velocity distributions of arbitrary shape.

For the case of compressible flows, several investigators have been interested in determining the basic nature of the effect of Mach number on the laminar separation point. Such methods as those of Howarth (ref. 4) and Cope and Hartree (ref. 5) provide means for determining the laminar separation point in a compressible flow. These methods are rather

cumbersome, however, and are not readily adaptable for rapidly estimating the effect of Mach number and velocity gradient on laminar separation. Stewartson (ref. 6) has recently shown that a vast simplification in the theory of compressible laminar boundary layers is possible if certain rather reasonable simplifying assumptions are made. On the basis of these assumptions, he was able to develop transforms which express the compressible laminar boundary layer in terms of an equivalent incompressible laminar boundary layer.

In the present paper the method of Von Doenhoff has been combined with Stewartson's transforms to provide a means for rapidly estimating the separation point of a laminar boundary layer in a compressible flow. The method developed is generally applicable to any two-dimensional flow in which the classical boundary-layer assumptions are satisfied. With the use of the method developed, the effect of Mach number on the laminar separation point has been calculated for a wide range of Mach number and velocity gradient. The results of these calculations are presented.

SYMBOLS

c	reference length
x	abscissa
y	ordinate
L	equivalent flat-plate length
a	speed of sound
U	velocity just outside boundary layer
u	velocity inside boundary layer
M	Mach number
F	nondimensional velocity gradient, $\dfrac{L}{U_0}\dfrac{dU}{dx}$
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
γ	ratio of specific heats, c_p/c_v
μ	viscosity

ρ . mass density

ν kinematic viscosity

k coefficient of heat conduction

Subscripts:

o reference condition taken at point of maximum velocity

1 any point along body just outside boundary layer

c compressible

i incompressible

ANALYSIS

Stewartson's Transforms

Stewartson (ref. 6) showed that the equations governing the behavior of two-dimensional laminar boundary layers in a compressible flow are identical to those governing their behavior in an incompressible flow if the variables in the two planes are related by the following transformations:

$$y_i = \frac{a_1}{a_0\sqrt{\nu_0}} \int_0^{y_c} \frac{\rho}{\rho_0}\, dy_c \tag{1}$$

$$x_i = \int_0^{x_c} \left(\frac{a_1}{a_0}\right)^{\frac{3\gamma-1}{\gamma-1}} dx_c \tag{2}$$

$$U_{i_1} = \frac{a_0}{a_1} U_{c_1} \tag{3}$$

where x_c, y_c, U_c and x_i, y_i, U_i are the coordinates and the velocity
just outside the boundary layer in the compressible and incompressible
planes, respectively. The subscript o refers to some convenient refer-
ence condition and the subscript 1 refers to conditions at a variable
point just outside the boundary layer. The detailed derivation of the
incompressible-boundary-layer equation from the compressible equations
with the use of transforms (1) to (3) is treated in reference 6.

The transformation of the compressible-boundary-layer equation to
an equivalent incompressible-boundary-layer equation with the use of
transforms (1) to (3) depends upon the following assumptions: The
boundary is thermally insulated, the viscosity is proportional to the
absolute temperature, and the Prandtl number $c_p\mu/k$ of the fluid is
unity. By assuming the boundary thermally insulated, the basic problem
of motion of the compressible boundary layer is isolated from extraneous
problems associated with heat transfer through the boundary surface.
The assumption that the Prandtl number is unity means that the stagna-
tion temperature is reached at the wall. The Prandtl number for air,
however, is actually about 0.715 (ref. 4); therefore, the temperature
near the wall is lower than that predicted. Since the viscosity
increases with the temperature, the assumption that the Prandtl number
is unity would indicate that the effects of viscosity are overestimated.
With regard to the remaining assumption that the viscosity is proportional
to the absolute temperature, experiments have shown that the viscosity
for air varies as the eight-ninths power of the absolute temperature
between 90° and 300° Kelvin (ref. 4). Thus, this assumption also results
in an overestimation of the effects of viscosity.

Von Doenhoff's Method

Since the transforms (1) to (3) permit the expression of the com-
pressible laminar boundary layer in terms of an equivalent incompressible
laminar boundary layer, it is apparent that all methods for calculating
the laminar separation point in incompressible flow are equally applicable
in the case of compressible flow. The laminar separation point is
defined as the point along the surface at which $\frac{\partial u}{\partial y} = 0$ at $y = 0$. Of
all methods available for calculating the laminar separation point in
incompressible flow, that due to Von Doenhoff (ref. 3) is perhaps the
simplest over-all method to apply and is used herein as a basis for
developing a method for rapidly estimating the laminar separation point
in a compressible flow.

With the use of the theory developed by Von Kármán and Millikan for
computing the separation point of a laminar boundary layer, Von Doenhoff
calculated the laminar separation point for a series of velocity dis-
tributions which consisted of a region of uniform velocity (flat-plate

flow) followed by a region of linearly decreasing velocity. These
calculations were generalized to the case of bodies with velocity dis-
tributions of arbitrary shape on the basis of the fundamental assump-
tion that the shape of the boundary-layer velocity distribution at the
point along a body at which the outside velocity is a maximum, that is,
at the point of application of the adverse velocity gradient, is very
nearly the same as that of a Blasius flat-plate distribution. Thus,
the condition of the boundary layer at the point of application of the
adverse velocity gradient can be represented by the flow over an equiva-
lent length of flat plate with a uniform velocity equal to the maximum
velocity. The assumption was also made that, for purposes of estimating
the laminar separation point, any velocity gradient likely to be encoun-
tered could be approximated by a straight line. Von Doenhoff presented
the results of his calculations in the form of the decrement in velocity
necessary to cause separation $\Delta U_{i_1}/U_{i_0}$ as a function of the nondimen-

sional velocity gradient $F_i = \dfrac{L_i}{U_{i_0}} \dfrac{dU_{i_1}}{dx_i}$, where L_i is the equivalent

flat-plate length, U_{i_0} is the maximum velocity, and dU_{i_1}/dx_i is the
velocity gradient. The relationship between $\Delta U_{i_1}/U_{i_0}$ and F_i given
in reference 3 is presented herein as figure 1. The equivalent length
of flat plate corresponding to the flow at the point of application of
the adverse velocity gradient can be found with the use of the following
equation (ref. 3, eq. (1)):

$$\frac{L_i}{c_i} = \int_0^{x_{i_0}/c_i} \left(\frac{U_{i_1}}{U_{i_0}}\right)^{8.17} d\left(\frac{x_i}{c_i}\right) \tag{4}$$

where U_{i_1}/U_{i_0} is the velocity ratio at any point on the body and the
integration is carried from the beginning of the flow to the point of
application of the adverse gradient. Equation (4) was obtained from
an integration of the Von Kármán momentum relation by making the
assumption that the boundary-layer velocity distribution at every point
along the body in the region of accelerating flow is the same as a
Blasius flat-plate distribution. Calculations of the laminar separa-
tion point on the NACA 0012 airfoil at zero lift made by the use of
Von Doenhoff's rapid method and calculations made by the use of the more
elaborate Von Kármán-Millikan theory are shown in reference 3 to agree
very closely.

Application of Stewartson's Transforms to Von Doenhoff's Method

In order to find the position on a body at which laminar separation occurs, the distribution of velocity outside the boundary layer and the Mach number at some typical point must be known. With this information, the laminar separation point can be calculated through the direct application of transforms (2) and (3) and Von Doenhoff's relation given in figure 1. Although relatively easy to apply, this method does involve a rather tedious point-by-point transformation from the compressible to the equivalent incompressible plane.

For this reason, a more rapid procedure has been developed for estimating the separation point of a compressible laminar boundary layer. This procedure consists essentially in the direct application of Von Doenhoff's method to the compressible velocity distribution, the effect of Mach number being accounted for by a simple multiplying factor applied to the measured velocity gradient in the compressible plane.

Consider the velocity gradient dU_c/dx_c in the compressible plane. From equation (2) it can be seen that $dx_i = \left(\dfrac{a_1}{a_0}\right)^{\frac{3\gamma-1}{\gamma-1}} dx_c$ and equation (3) states that $U_{i_1} = \dfrac{a_0}{a_1} U_{c_1}$. Consequently, the compressible and incompressible velocity gradients are related as follows:

$$\frac{dU_{i_1}}{dx_i} = \frac{d\left(\frac{a_0}{a_1} U_{c_1}\right)}{dx_c}\frac{dx_c}{dx_i} = \frac{d\left(\frac{a_0}{a_1} U_{c_1}\right)}{dx_c}\;\frac{1}{\left(\frac{a_1}{a_0}\right)^{\frac{3\gamma-1}{\gamma-1}}}$$

or

$$\frac{dU_{i_1}}{dx_i} = \left[U_{c_1}\frac{d\left(\frac{a_0}{a_1}\right)}{dx_c} + \frac{a_0}{a_1}\frac{dU_{c_1}}{dx_c}\right]\;\frac{1}{\left(\frac{a_1}{a_0}\right)^{\frac{3\gamma-1}{\gamma-1}}}$$

If the reference velocity of sound a_0 is taken at the point of maximum velocity U_{c_0}, that is, at the point of application of the adverse velocity gradient, then $U_{c_0} = U_{i_0}$ and

$$\frac{d\left(\frac{U_{i_1}}{U_{i_0}}\right)}{dx_i} = \left[\frac{U_{c_1}}{U_{c_0}}\frac{d\left(\frac{a_0}{a_1}\right)}{dx_c} + \frac{a_0}{a_1}\frac{d\left(\frac{U_{c_1}}{U_{c_0}}\right)}{dx_c}\right]\frac{1}{\left(\frac{a_1}{a_0}\right)^{\frac{3\gamma-1}{\gamma-1}}} \tag{5}$$

By the use of the one-dimensional energy equation, the terms a_1/a_0 and $d\left(\frac{a_0}{a_1}\right)/dx_c$ can be written in terms of the Mach number M_0 and the velocity ratio U_{c_1}/U_{c_0}:

$$\frac{a_1}{a_0} = \sqrt{1 + \frac{\gamma - 1}{2}M_0{}^2\left[1 - \left(\frac{U_{c_1}}{U_{c_0}}\right)^2\right]} \tag{6}$$

and

$$\frac{d\left(\frac{a_0}{a_1}\right)}{dx_c} = \frac{\frac{\gamma - 1}{2}M_0{}^2\frac{U_{c_1}}{U_{c_0}}\frac{d\left(\frac{U_{c_1}}{U_{c_0}}\right)}{dx_c}}{\left\{1 + \frac{\gamma - 1}{2}M_0{}^2\left[1 - \left(\frac{U_{c_1}}{U_{c_0}}\right)^2\right]\right\}^{3/2}} \tag{7}$$

An examination of equations (5) to (7) indicates that a linear velocity distribution in the compressible plane transforms into a nonlinear distribution in the equivalent incompressible plane. The basic assumption

is made, however, that the determination of the approximate laminar
separation point in the equivalent incompressible plane can be made
from a knowledge of the transformed adverse velocity gradient at the
point of maximum velocity. This assumption presupposes that there is
a discontinuity in the compressible velocity distribution at the point
of application of the adverse gradient or, at least, that the velocity
distribution can be approximated in this manner. On the basis of this
assumption, $\dfrac{a_1}{a_0} = 1.0$ and

$$\frac{d\left(\dfrac{a_0}{a_1}\right)}{dx_c} = \frac{\gamma - 1}{2} M_0^2 \frac{d\left(\dfrac{U_{c_1}}{U_{c_0}}\right)}{dx_c} \qquad (8)$$

Substitution of equation (8) into equation (5) and the value 1.0 for
the terms U_{c_1}/U_{c_0} and a_0/a_1 yields the following relation:

$$\frac{d\left(\dfrac{U_{i_1}}{U_{i_0}}\right)}{dx_i} = \left(1 + \frac{\gamma - 1}{2} M_0^2\right)\frac{d\left(\dfrac{U_{c_1}}{U_{c_0}}\right)}{dx_c} \qquad (9)$$

Thus, the value of the velocity gradient at the point of maximum veloc-
ity in the incompressible plane is obtained by multiplying the measured
value of the velocity gradient in the compressible plane by the fac-
tor $1 + \dfrac{\gamma - 1}{2} M_0^2$, where it should be remembered that M_0 is the Mach
number at the point of application of the adverse gradient in the com-
pressible plane. The factor $1 + \dfrac{\gamma - 1}{2} M_0^2$ is plotted as a function
of Mach number in figure 2.

The velocity gradient in the equivalent incompressible plane (eq. (9))
must be made nondimensional in terms of the equivalent flat-plate length
in the incompressible plane if it is to be used with the relation given
in figure 1 for finding the velocity recovery before laminar separation
occurs in an incompressible flow. The equivalent length of flat plate in
the incompressible plane can be determined from the relation

$$L_i = \int_0^{x_{i_0}} \left(\frac{U_{i_1}}{U_{i_0}}\right)^{8.17} dx_i \qquad (10)$$

or, in terms of variables in the compressible plane, from the relation

$$L_i = \int_0^{x_{c_0}} \left(\frac{U_{c_1}}{U_{c_0}}\right)^{8.17} \left(\frac{a_0}{a_1}\right)^{8.17} \left(\frac{a_1}{a_0}\right)^{\frac{3\gamma-1}{\gamma-1}} dx_c \qquad (11)$$

For air, $\gamma = 1.4$; therefore, $\dfrac{3\gamma - 1}{\gamma - 1} = 8.0$ and equation (11) reduces to

$$L_i = \int_0^{x_{c_0}} \left(\frac{U_{c_1}}{U_{c_0}}\right)^{8.17} \left(\frac{a_0}{a_1}\right)^{0.17} dx_c \qquad (12)$$

Equation (12) can be used directly with the compressible velocity distribution to determine L_i. In many cases, however, it would appear that the value of $\left(\dfrac{a_0}{a_1}\right)^{0.17}$ would be so near unity as to have a negligible effect on L_i. In this case, the equivalent flat-plate length L_i can be determined from the equation

$$L_i = \int_0^{x_{c_0}} \left(\frac{U_{c_1}}{U_{c_0}}\right)^{8.17} dx_c \qquad (13)$$

The complete expression for the nondimensional velocity gradient in the equivalent incompressible plane in terms of variables in the compressible plane is therefore

$$\frac{L_i}{U_{i_0}} \frac{dU_{i_1}}{dx_i} = \left(1 + \frac{\gamma - 1}{2} M_0^2\right)\frac{L_i}{U_{c_0}} \frac{dU_{c_1}}{dx_c} \tag{14}$$

In the determination of L_i and $d\left(\frac{U_{c_1}}{U_{c_0}}\right)\Big/dx_c$ the value of x_c can, of course, be expressed in terms of a reference length in the compressible plane.

The determination of the laminar separation point corresponding to a known velocity distribution at a given Mach number involves the following steps:

(a) Graphical measurement of the velocity gradient dU_c/dx_c in the compressible plane. This process may require fairing of the given velocity distribution in such a way that the adverse velocity gradient begins at a distinct point and is approximated by a straight line.

(b) Calculation of the equivalent flat-plate length by the use of equation (13).

(c) Determination of the factor $1 + \frac{\gamma - 1}{2} M_0^2$ from figure 2.

The nondimensional velocity gradient in the equivalent incompressible plane may then be determined by the use of equation (14). The velocity ratio corresponding to laminar separation in the incompressible plane may be found from the curve of figure 1 which expresses the velocity decrement necessary to cause laminar separation in terms of the nondimensional velocity gradient. The velocity ratio corresponding to laminar separation in the compressible plane is found with the use of the relation $\frac{U_{i_1}}{U_{i_0}} = \frac{a_0}{a_1} \frac{U_{c_1}}{U_{c_0}}$. The relationship between U_{i_1}/U_{i_0} and U_{c_1}/U_{c_0} for different Mach numbers is plotted in figure 3.

In some cases, considerable fairing of the given velocity distribution may be required in order to obtain what appears to be a reasonable measure of $d\left(\frac{U_{c_1}}{U_{c_0}}\right)\Big/dx_c$. Under such circumstances, the position of the calculated laminar separation point with respect to the given velocity distribution may indicate that a somewhat different fairing of the

velocity distribution would give a more realistic approximation of that portion of the velocity gradient along which laminar separation occurs. A second calculation of the laminar separation point may then be indicated.

Comparison With Results of Previous Analyses

The effect of Mach number on the position at which laminar separation occurs in a boundary-layer flow progressing against a linearly decreasing velocity from the leading edge has been investigated by Howarth (ref. 4) who employed a Von Kármán-Pohlhausen type of analysis, and by Stewartson (ref. 6) who applied his transforms to the incompressible solution developed by Howarth in 1938 (ref. 1). This type of velocity distribution has a value of F_c equal to zero since the equivalent length of flat plate is zero. In order to provide a comparison between the method of the present investigation and those used in references 4 and 6, the laminar separation point has been calculated by the method presented herein for the case of F_c equal to zero for a range of Mach number. The results of these computations are presented in figure 4 together with the results obtained by Howarth and Stewartson. Examination of figure 4 indicates that considerable difference exists at all Mach numbers between the predictions of the three methods. For zero Mach number, the value of the decrement in velocity ratio of 0.12 given by Howarth in reference 1 is probably the most exact. The magnitude of the differences between the three methods, however, appears to be roughly the same for all Mach numbers. This result can be seen more clearly in figure 5 in which the difference in $\Delta U_{c_1}/U_{c_0}$ at Mach number zero and at some arbitrary Mach number is plotted against Mach number. The trends shown in figure 5 indicate that the predicted effect of Mach number on the laminar separation point as determined by the method of the present investigation is in substantial agreement with the results obtained by Stewartson and Howarth for the case of linearly decreasing velocity from the leading edge.

EFFECT OF MACH NUMBER ON THE LAMINAR SEPARATION POINT

In order to show the effect of Mach number on the position of laminar separation for a series of values of F_c, the nondimensional compressible velocity gradient, calculations have been made for Mach numbers varying from 0 to 10 and for nondimensional compressible velocity gradients F_c varying from 0 to -0.12. These calculations are for velocity distributions represented by a region of uniform velocity equal to the maximum velocity followed by a region of uniformly decreasing velocity. The results of these calculations are presented in figure 6

and show that the velocity recovery obtainable before laminar separation decreases as the Mach number increases for all nondimensional velocity gradients. For a given velocity gradient, therefore, the laminar separation point moves forward as the Mach number increases. Figure 6 also shows that the shape of the curves of velocity recovery against Mach number is not influenced to any large extent by the value of the velocity gradient.

It is important to point out that the individual curves presented in figure 6 show the effect on the laminar separation point of increasing the Mach number for fixed values of the velocity gradient. The problem of most practical interest, however, is that of a body having fixed geometry. In this case, the velocity gradients vary as the Mach number varies. Thus, when the laminar separation point on a given body for different Mach numbers is calculated, the value of the compressible velocity gradient F_c varies with Mach number. In general, adverse velocity gradients become steeper as the Mach number increases so that, for a given body, the forward movement of the laminar separation point with increasing Mach number would be greater than is indicated in figure 6 for a given velocity gradient.

A word of caution should be added with regard to the type of flow field to which the method developed herein should be applied. It will be recalled that the method developed is based on the classical boundary-layer assumptions. These assumptions state that the rate of change of velocity in the x-direction must be small with respect to its rate of change in the y-direction. The method obviously cannot be applied in the vicinity of a discontinuity in velocity (pressure) such as that associated with the presence of a shock wave.

CONCLUDING REMARKS

A method has been developed for rapidly estimating the separation point of a laminar boundary layer in a compressible flow. The method consists of an extension of Von Doenhoff's simplified solution for the incompressible case (NACA TN 671) and makes use of a set of transforms derived by Stewartson (Proc. Roy. Soc., 1949) which permit, under certain assumptions, the expression of compressible laminar layers in terms of equivalent incompressible laminar layers. The method developed is generally applicable to any two-dimensional flow in which the classical boundary-layer assumptions are satisfied. The dependence of the method upon the boundary-layer assumptions, of course, means that it should not be applied to determine whether the pressure rise through a shock wave causes separation.

Calculations of the laminar separation point for a wide range of Mach number and velocity gradient indicate that, for all velocity gradients, the amount of velocity recovery possible before laminar separation occurs decreases as the Mach number increases.

Langley Aeronautical Laboratory,
    National Advisory Committee for Aeronautics,
        Langley Field, Va., November 19, 1952.

REFERENCES

1. Howarth, L.: On the Solution of the Laminar Boundary Layer Equations. Proc. Roy. Soc. (London), ser. A, vol. 164, no. 919, Feb. 18, 1938, pp. 547-579.

2. Von Kármán, Th., and Millikan, C. B.: On the Theory of Laminar Boundary Layers Involving Separation. NACA Rep. 504, 1934.

3. Von Doenhoff, Albert E.: A Method of Rapidly Estimating the Position of the Laminar Separation Point. NACA TN 671, 1938.

4. Howarth, L.: Concerning the Effect of Compressibility on Laminar Boundary Layers and Their Separation. Proc. Roy. Soc. (London), ser. A, vol. 194, no. A 1036, July 28, 1948, pp. 16-42.

5. Cope, W. F., and Hartree, D. R.: The Laminar Boundary Layer in Compressible Flow. Phil. Trans. Roy. Soc. (London), ser. A, vol. 241, June 22, 1948, pp. 1-68.

6. Stewartson, K.: Correlated Incompressible and Compressible Boundary Layers. Proc. Roy. Soc. (London), ser. A, vol. 200, June 30, 1949, pp. 84-100.
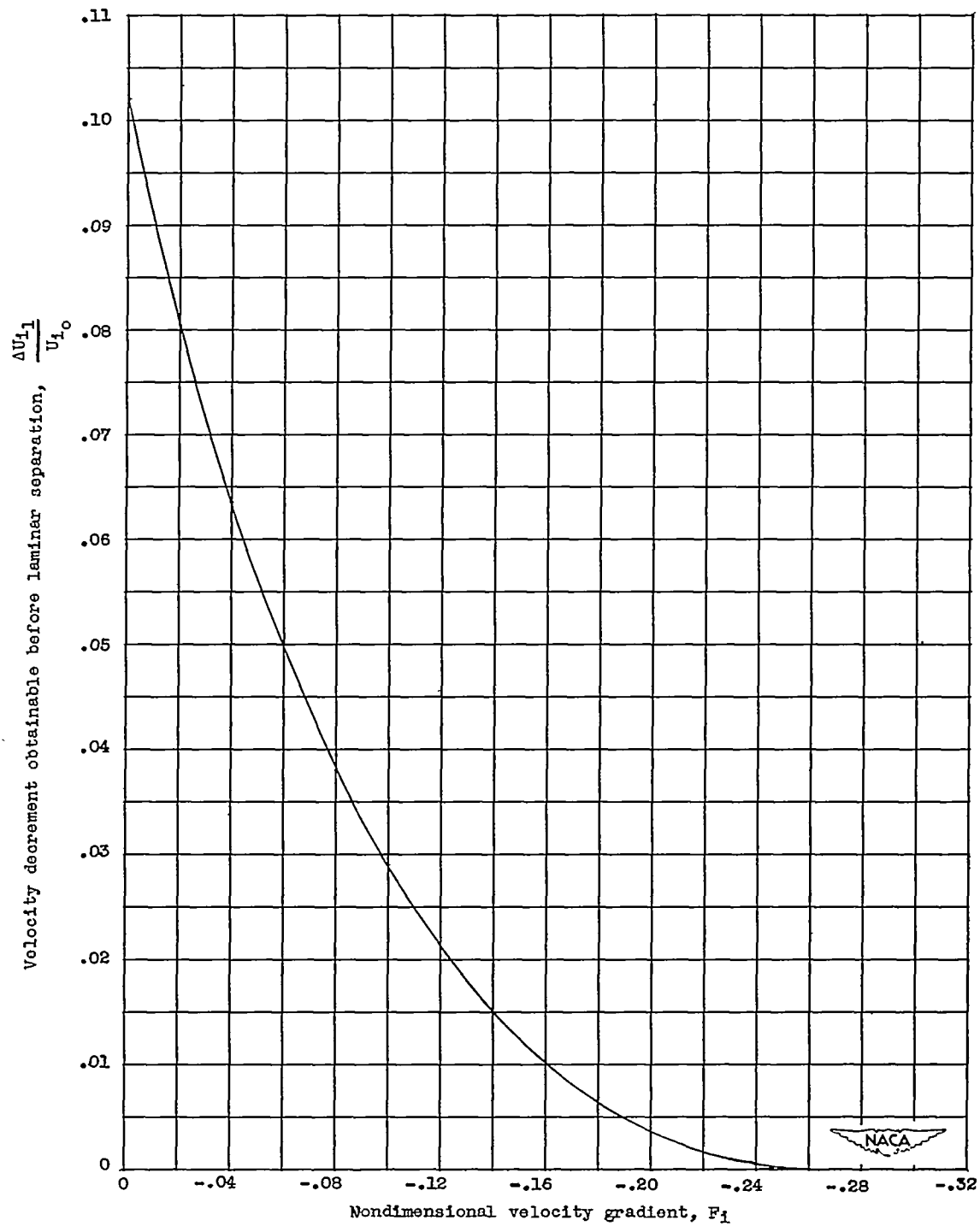
Figure 1.- Incompressible velocity decrement $\Delta U_{1_1}/U_{1_0}$ obtainable before laminar separation occurs as a function of the velocity gradient F_1 (ref. 3).
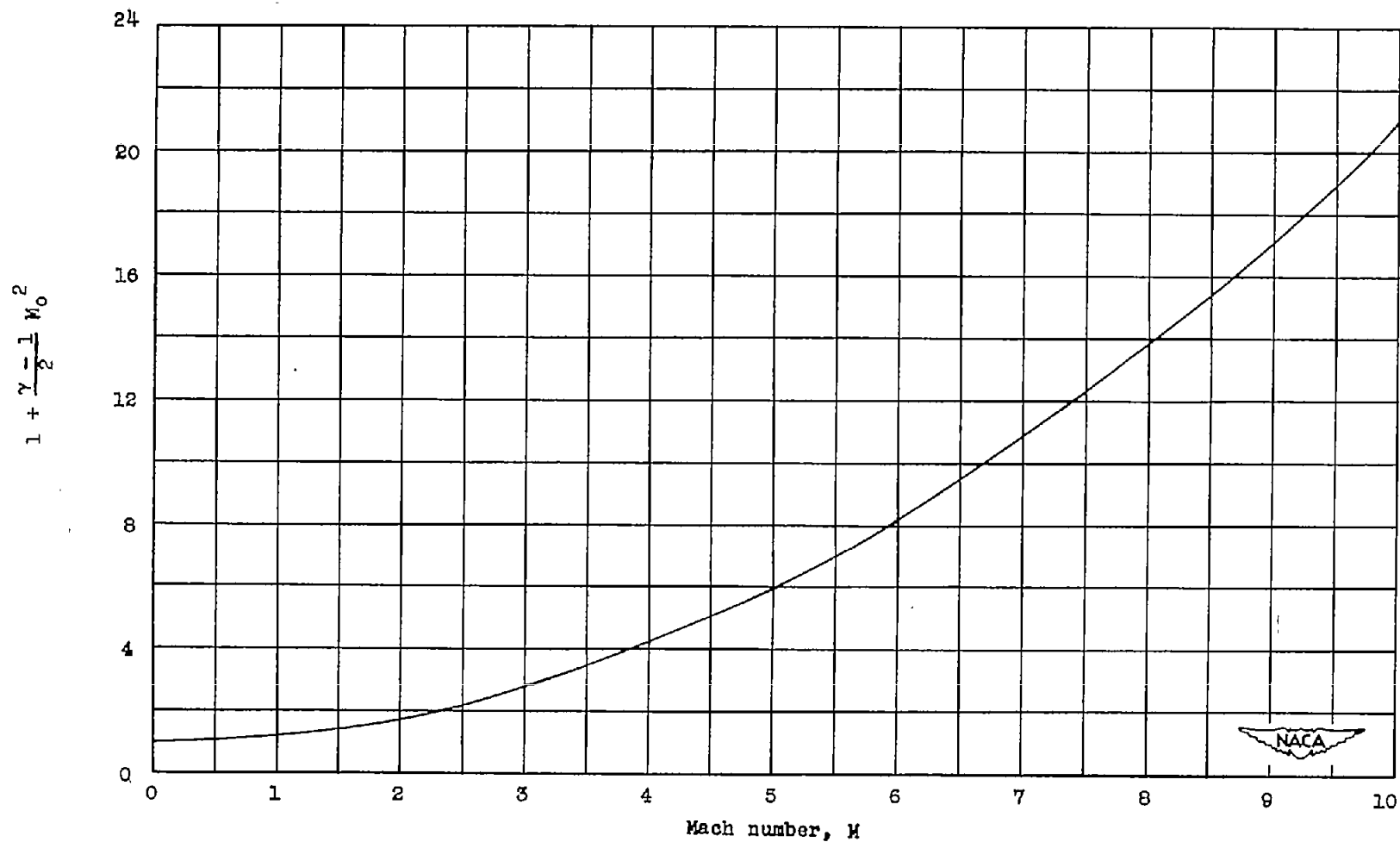
Figure 2.- The quantity $1 + \dfrac{\gamma - 1}{2} M_0^2$ as a function of Mach number.
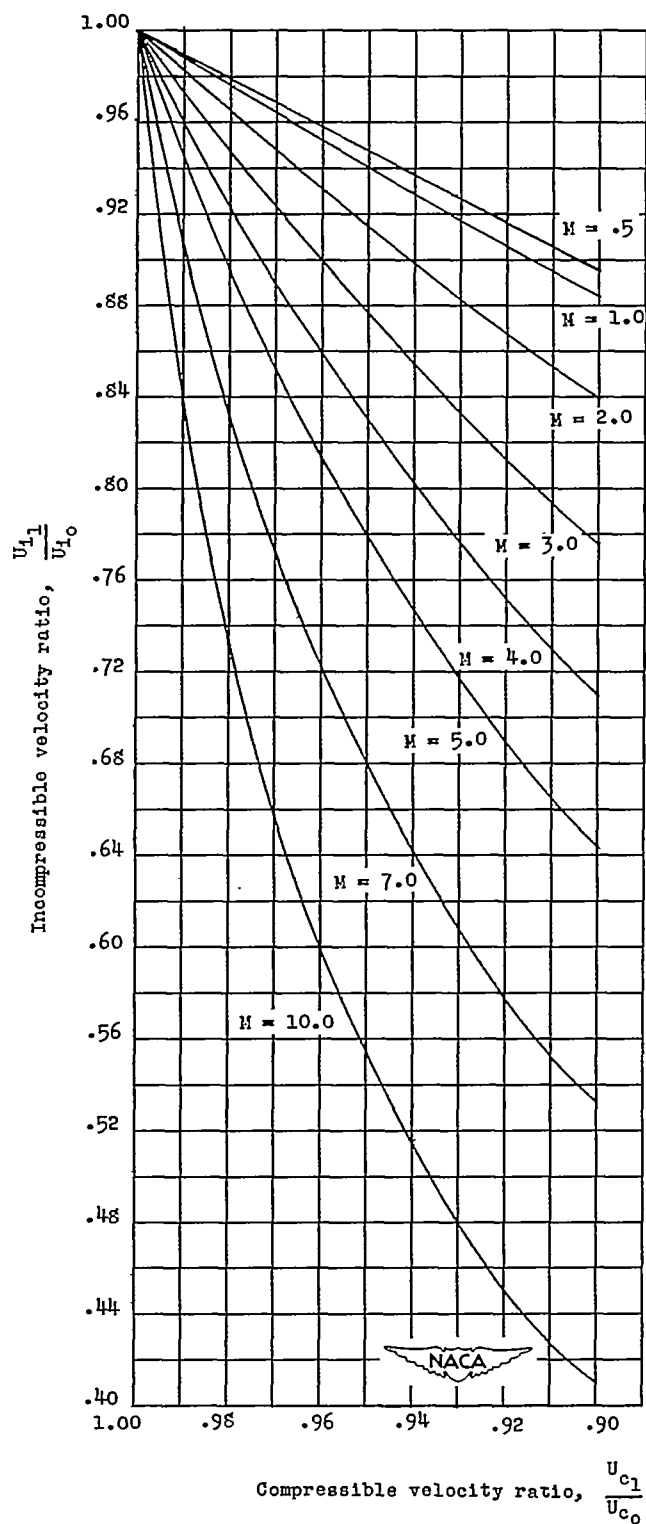
Figure 3.- Incompressible velocity ratio as a function of compressible
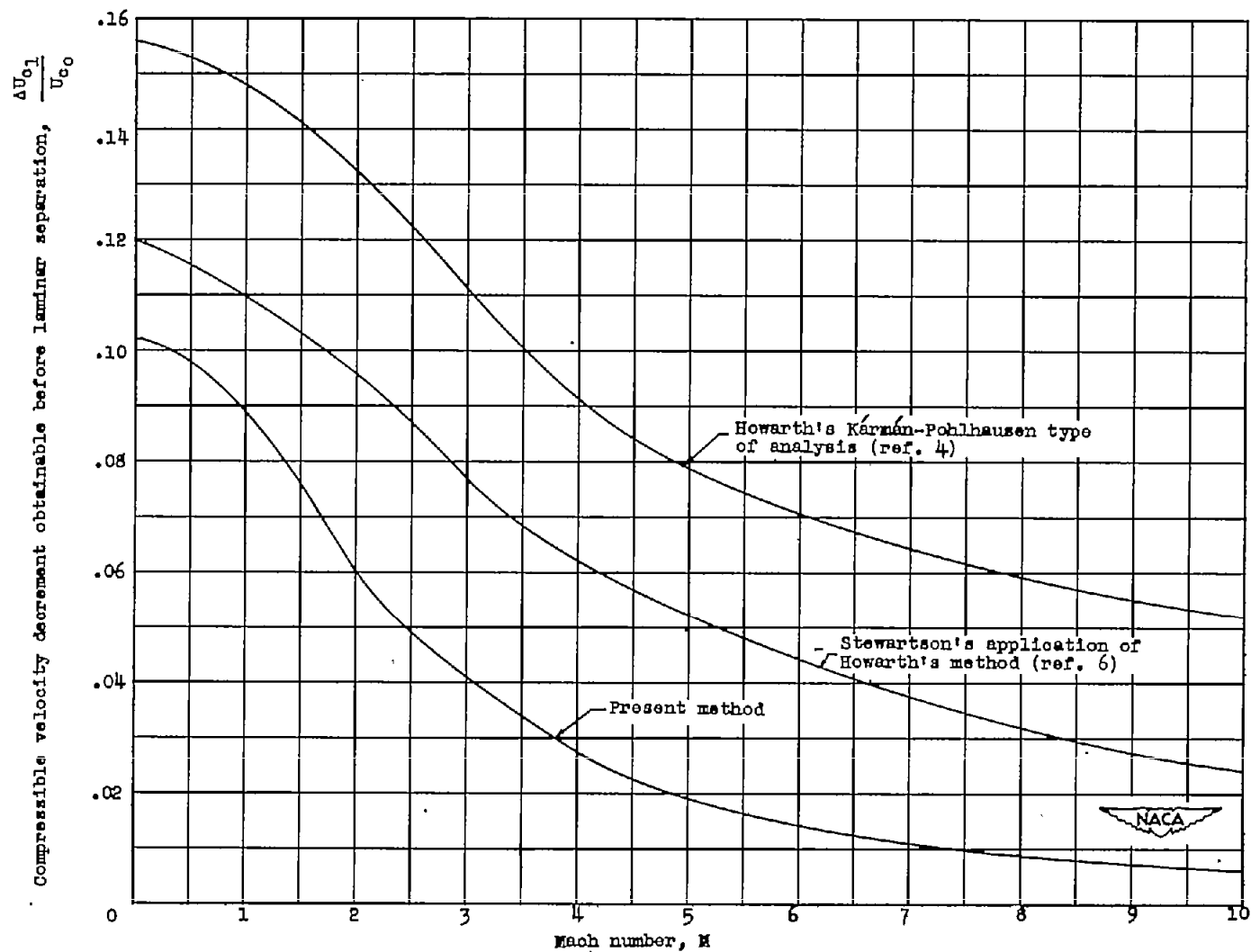velocity ratio for different Mach numbers.

Figure 4.- Comparison of the results of the present method with those
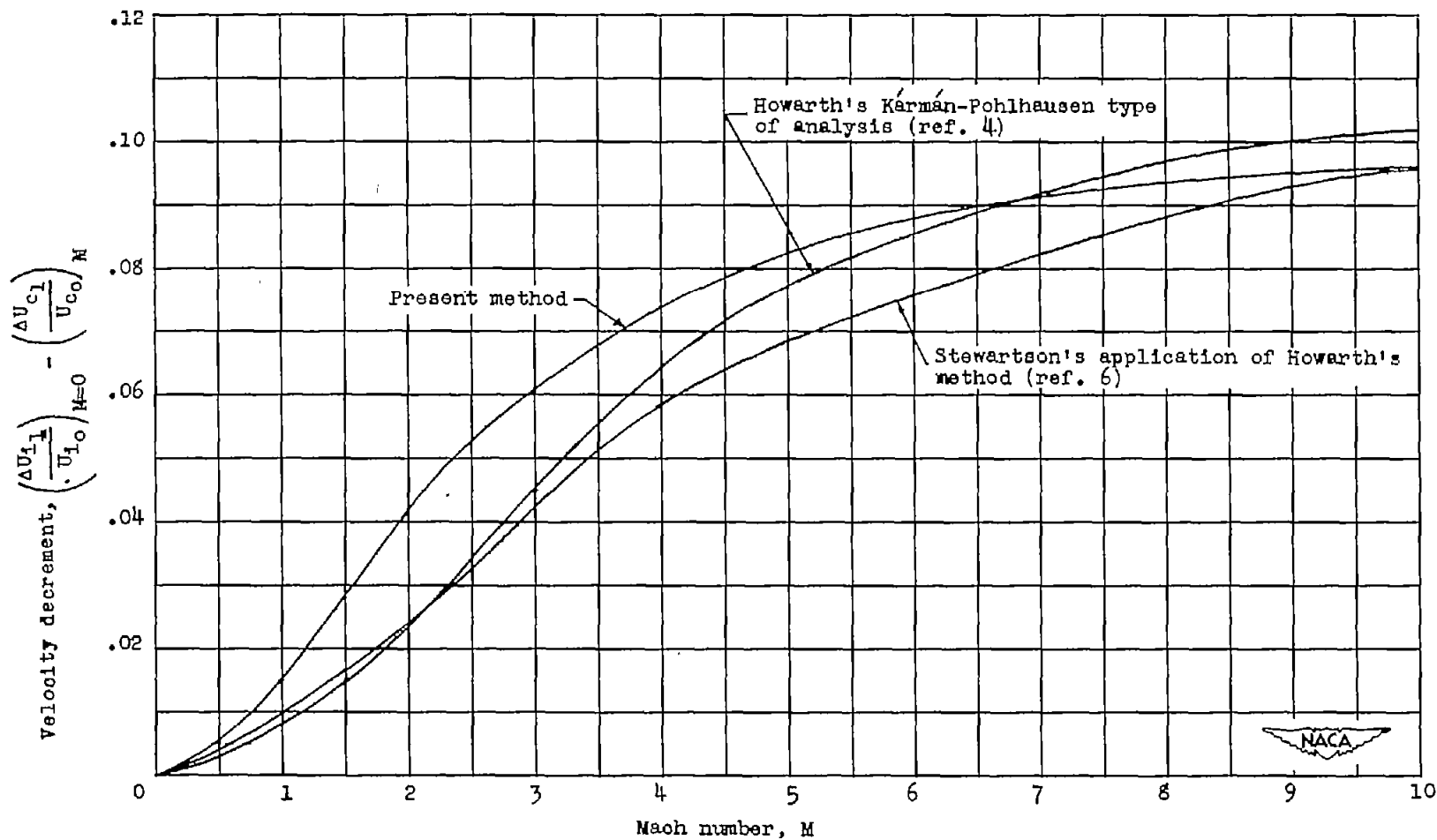of Howarth and Stewartson for the case when F = 0.

Figure 5.- Comparison of effect of Mach number on laminar separation
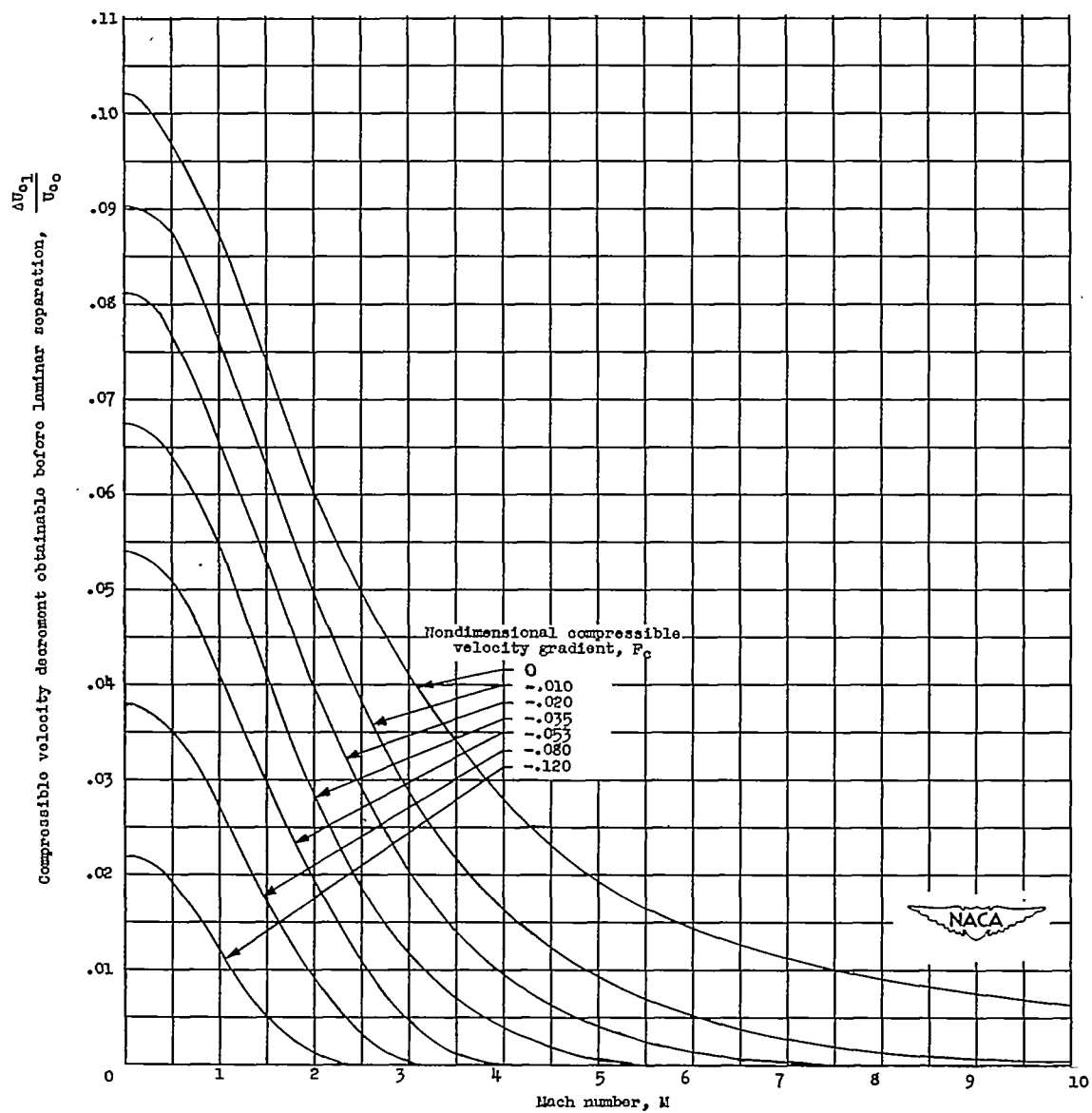as predicted by the present method and the methods of Howarth and
Stewartson for F = 0.

Figure 6.- Compressible velocity decrement $\Delta U_{c_1}/U_{c_0}$ obtainable before separation as a function of Mach number for various velocity gradients F_c.